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## Rocket Science without the Chemistry

### Introduction

It's quite remarkable how much analysis of rocket
motion can be done when one is armed with some 1
physics, a bit of mathematics and a spread sheet.

A simple rocket system that converts plastic bottles to rockets can be bought relatively cheaply. A good example is the aquapod®, upon which the system in the photograph is based. It's an upside-down, pressurised plastic bottle, half filled with water. When
released, water is ejected at high speed from its tail 2
end resulting in the rocket accelerating upwards. Because downward momentum is given to the water, the rocket gains upward momentum. Early on in the investigations into these rockets, it was discovered that a ball placed on top of the bottle was far easier to investigate and model.

**Diagram 1**

### Energy Analysis Considering Gravity

The simplest possible analysis that can be done to the rocket is to apply conservation of energy. We could approximate that the work done in compressing the gas eventually becomes gravitational potential energy of the ball. How much energy is stored in the compressed gas? For this, we need a bit of A level Maths in order to calculate the area below an isothermal
compression in a $p$-$V$ diagram. This is the result of the integration (that doesn't require learning). 3

$$W = p \times V \times \ln\left(\frac{p}{p_{\text{atm}}}\right) \qquad \textbf{Equation 1}$$

where $p$ is the high pressure inside the bottle, $V$ is the volume of compressed air inside the bottle and $p_{\text{atm}}$ is atmospheric pressure.

So a 2 litre bottle half filled with water and with a pressure of $4.4 \times 10^5$ Pa inside it can supply around 650 J of energy. If all this energy were transferred to a 0.45 kg ball, the ball should
attain a height of some 150 m. In practice, 4
however, only a small fraction of this energy is transferred to the ball.

This experiment was carried out with a set of plastic bottles varying in volume from 500 ml to 3.0 litre. Each bottle was half full of water
at take-off and each bottle was pumped to 5
a pressure of $4.4 \times 10^5$ Pa (from an initial atmospheric pressure of $1.0 \times 10^5$ Pa). The results are shown in the graph (diagram 2).

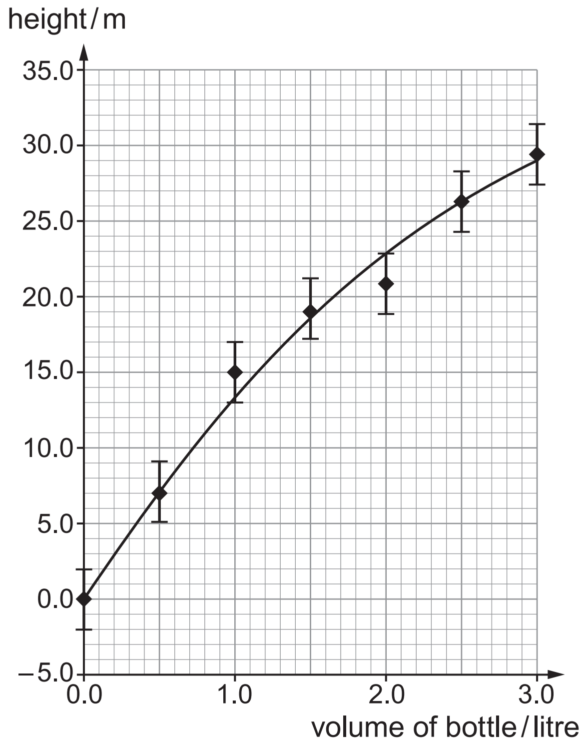


**Diagram 2**

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Unsurprisingly perhaps, this simplistic conservation of energy argument has not been successful and the relationship between maximum height of the rocket and volume of the bottle is not directly proportional as was predicted by conservation of energy. 6

**Ideal Rocket Theory (fixed exhaust speed and ignoring gravity)**

Now for some more detailed analysis. We need to look at what causes the acceleration of the rocket. In effect, we need to apply Newton's second law to the rocket and to do that we need to know the rate of change of momentum of water leaving the bottle. First, let's define some terms: 7

$m_0$ = total initial mass of the rocket;
$t$ = time counting from blast-off;
$\frac{\Delta m}{\Delta t}$ = constant rate of ejecting of mass; 8
$u$ = constant speed of the water leaving the bottle (relative to the rocket).

The resultant force exerted on the water is equal to its rate of change of momentum. The momentum gained by the water per second is $u\frac{\Delta m}{\Delta t}$ (remember, $\frac{\Delta m}{\Delta t}$ is the mass leaving the bottle per second). This, therefore, is the force experienced by the water and by Newton's 3rd law, this is also the force experienced by the rocket. So, we now know that the thrust force acting on the rocket is $u\frac{\Delta m}{\Delta t}$. 9

The mass of the rocket is decreasing at a constant rate of $\frac{\Delta m}{\Delta t}$ so its mass at any time $t$ is given by $\left(m_0 - \frac{\Delta m}{\Delta t}t\right)$. 10

This is enough information to use Newton's 2nd law ($F = ma$) to give an equation for the acceleration. Mathematics (that won't require learning) then leads to solutions for both velocity and height. Here are the equations: 11

$$v = -u\ln(1 - \alpha t) \qquad \textbf{Equation 2}$$
$$h = \frac{u}{\alpha}[(1-\alpha t)\ln(1-\alpha t) + \alpha t] \qquad \textbf{Equation 3}$$

$\alpha$ is the ratio of the rate of loss of mass to the initial mass, $\left(\alpha = \dfrac{\frac{\Delta m}{\Delta t}}{m_0}\right)$.

We can now try to apply these equations to a typical 2 litre bottle. The total mass of the water, bottle and ball is around 1.5 kg (i.e. $m_0$ = 1.5 kg). Of this, 1.0 kg is water, 0.45 kg for the ball and the bottle has a mass of 0.05 kg. From high speed video analysis of the rocket, all 1.0 kg of the water is expelled in 0.14 s so that we can calculate the mean rate of decrease of mass of the rocket: 12

$$\frac{\Delta m}{\Delta t} = k = \frac{1.0}{0.14} = 7.1\,\text{kg}\,\text{s}^{-1}$$



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We can also use these figures to determine the exhaust speed of the water when we know that the radius of the bottle neck is 1.1 cm.

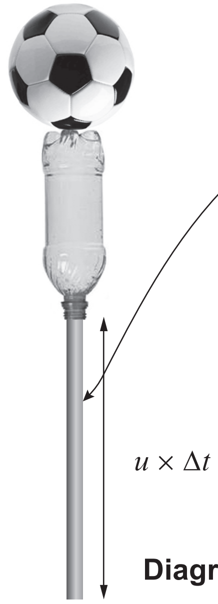


**Diagram 3**

The mass of the water ($\Delta m$) in the cylinder of length $u \times \Delta t$ is:

$$\text{volume} \times \text{density} = \pi r^2 \times u \times \Delta t \times \rho$$

which gives us a relationship between the rate of mass loss and the exhaust velocity:

$$\frac{\Delta m}{\Delta t} = \pi r^2 \rho u \qquad \textbf{Equation 4}$$ 13

When this expression is equated to the actual rate of loss of mass (7.1 kg s$^{-1}$), it gives a value of $u$ of around 19 m s$^{-1}$ (remembering that the density of water is rather conveniently 1 000 kg m$^{-3}$).

**Comparison Between Ideal Rocket Theory and Experiment**

This is enough information to put into the rocket equations (2 & 3) and compare with the motion of an actual rocket. The easiest way of doing this is to use a spreadsheet - we can enter both the rocket equations and actual data and compare the two. In the graph shown (diagram 4), the actual 2 litre rocket data is shown as a continuous line. The theoretical rocket equation is
represented by the dotted lines. The data 14
for the height of the actual rocket was gathered by using a comparatively cheap digital camera set to 220 frames per second. The video of the rocket motion was then analysed frame by frame using a 30 cm ruler to measure distances on the screen and the continuous curve in the graph obtained.

Interestingly, when these data and the rocket equations were put into a spreadsheet, the value of exhaust speed ($u$) of
19 m s$^{-1}$ did not produce an ideal fit (see 15
diagram 4). The best fit for the early motion of the rocket was provided by an exhaust speed of around 21 m s$^{-1}$ whereas the later motion of the rocket fits better with an exhaust speed of around 18 m s$^{-1}$. This may seem like a bad agreement but, on the other hand, these discrepancies could be pointing toward the reason for the disagreement.

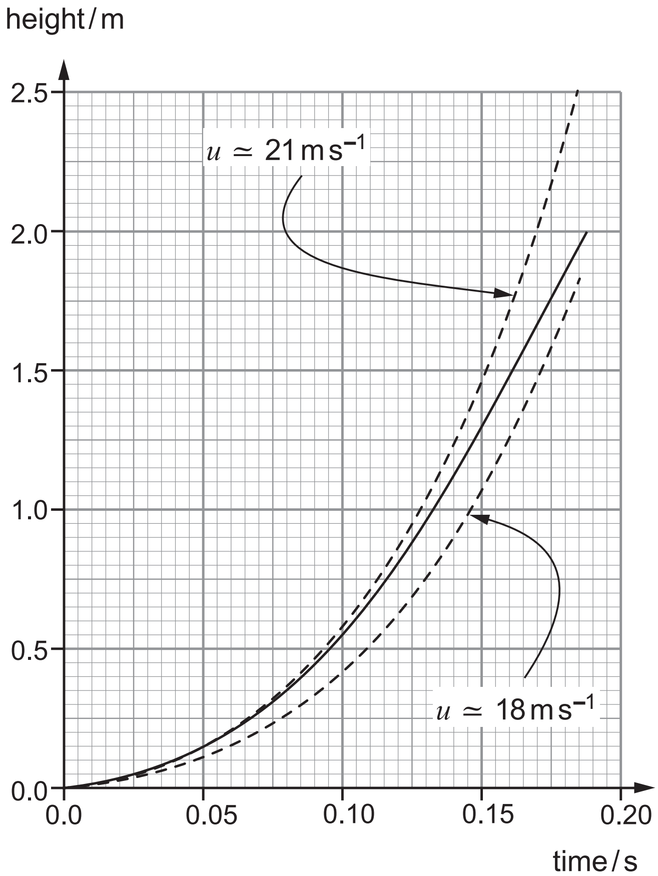


**Diagram 4**

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**Rocket Theory with Decreasing Pressure and Exhaust Speed**

The rocket equation fits well for the first 0.10 s of its flight with an exhaust speed of around 21 m s$^{-1}$ but then the actual rocket does not keep up with its theoretical counterpart and the actual rocket seems to fit better with an exhaust speed of around 18 m s$^{-1}$. What could be the reason for this? Put simply, it's the decrease in pressure of the air inside the bottle as the water is leaving. But how can we model the pressure inside the bottle? Simplistically, we can use Boyle's law. Robert Boyle in the mid 1600s said for a fixed mass of gas at constant temperature: 16

pressure × volume = constant

We know that the initial volume of the gas is around 1 litre (for the 2 litre bottle). The final volume of the gas rather obviously will be 2 litre. Boyle's law therefore tells us that the pressure at the start will be approximately double the final pressure. In between these two stages, each gramme of water that is expelled provides an extra 1 cm$^3$ of air in the bottle and the corresponding pressure drop can easily be calculated using Boyle's law. 17

Now that we have the details to model the pressure drop in the bottle, it is possible to calculate the speed of the water coming out of the bottle. All we have to do is use Bernoulli's equation.

$$p_{\text{atm}} = p - \tfrac{1}{2}\rho u^2 \qquad \textbf{Equation 5}$$

Surprisingly enough, this means that the exit speed of the water is independent of the size of the bottle opening! 18

$$u = \sqrt{\frac{2(p - p_{\text{atm}})}{\rho}} \qquad \textbf{Equation 6}$$

Now we can use this equation to calculate the exhaust speed of the water. The density of water ($\rho$) is 1 000 kg m$^{-3}$ and the initial ($p$–$p_{\text{atm}}$) was 3.4 × 10$^5$. This gives an initial exhaust speed of around 26 m s$^{-1}$. 19

For completion, gravity and air resistance should also be incorporated into our model. Gravity is easy enough but what about air resistance? A simple theory for air resistance is that the increase in air resistance is proportional to velocity squared. In fact, if we look up the air resistance of a sphere, we should find: 20

$$F_{\text{drag}} = 0.47 \times \tfrac{1}{2}\rho_{\text{air}} v^2 \times A \qquad \textbf{Equation 7}$$

where $A$ is the maximum cross-sectional area of the sphere, $\rho_{\text{air}}$ is the density of air and $v$ is the speed of the sphere.

Another great advantage of placing a football on top of the water bottle rocket is that the air resistance can be modelled based on the dimensions of the football. This assumes that the bottle underneath the football has no effect on the air resistance but should be a reasonable approximation considering that the cross-sectional area of the football is far greater than that of the bottle. The density of air ($\rho_{\text{air}}$) is 1.20 kg m$^{-3}$ and the diameter of the football is 22.0 cm and they can both be inputted into the air resistance equation. 21

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All this information should give us a final resultant force acting on the rocket of:

$$F_{\text{res}} = \pi r^2 \rho u^2 - mg - 0.0107v^2 \qquad \textbf{Equation 8}$$

where: $u$ = instantaneous exhaust speed of the water
$r$ = radius of the bottle opening 22
$\rho$ = density of water (1000 kg m$^{-3}$)
$m$ = instantaneous mass of the rocket (including the water and football)
$v$ = instantaneous speed of the rocket

**Final Comparison between Theory and Experiment**

When all this data is put into a spreadsheet with time going up in steps of 1/220th of a second (to match the digital camera) and all rocket data calculated for all the time intervals. This is the end result. 23

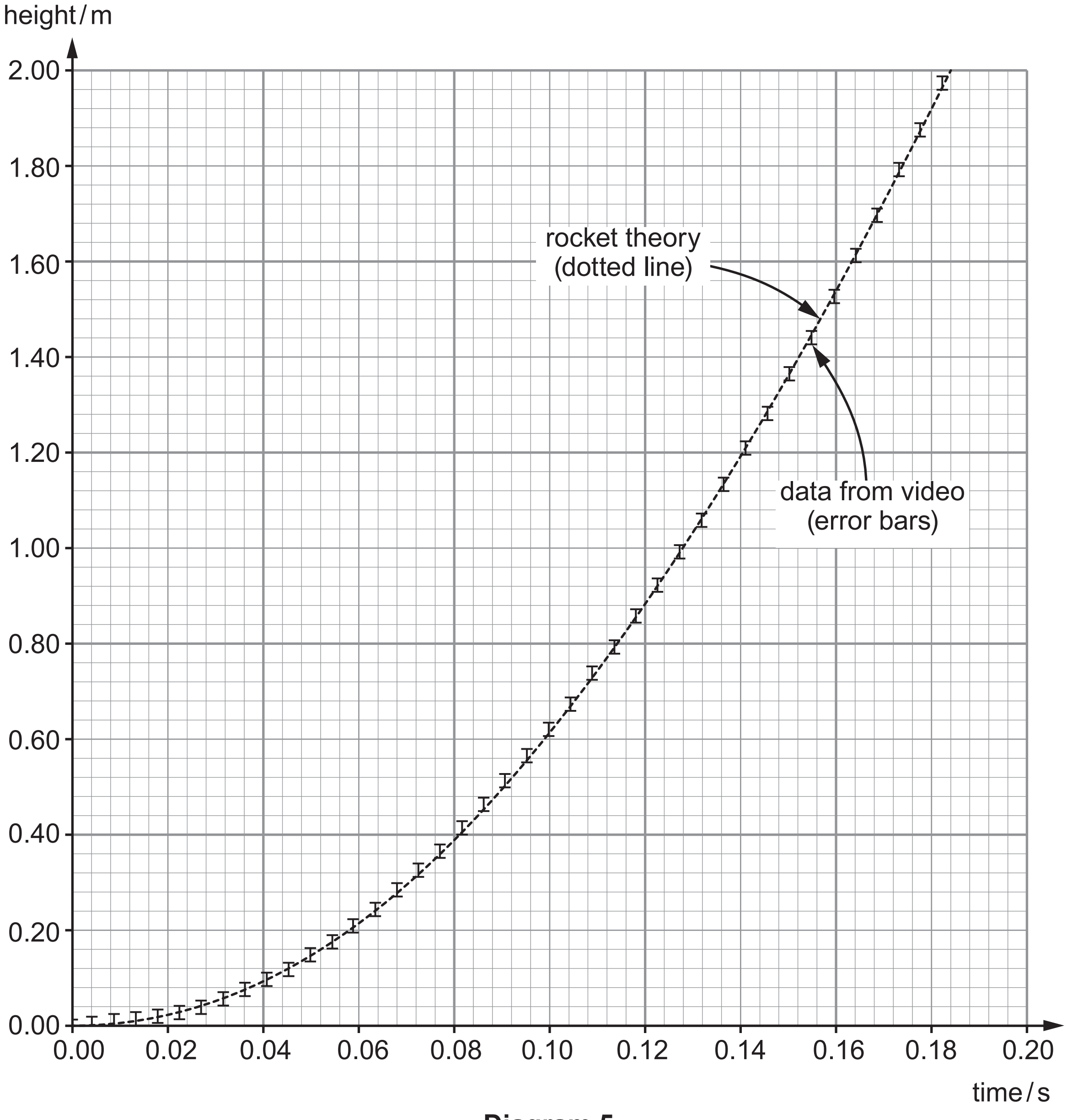


**Diagram 5**

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Here, the rocket theory (dotted line) is in excellent agreement with the experimental results (error bars). At no point is the computed model data outside the error bars corresponding to the actual motion of the rocket (these error bars simply correspond to $\pm 0.5$ mm reading from the ruler next to the computer screen). The final best fit parameters used were $m_0$ = 1.52 kg, initial pressure = $4.7 \times 10^5$ Pa and radius of bottle opening = 1.019 cm. 24

In conclusion, the motion of a plastic water bottle rocket has been analysed using purely A level Physics with a touch of Bernoulli's equation and drag theory. Although the mathematics used can be complicated, this is relatively easily remedied by using numerical methods in a computer spreadsheet. The results are astonishingly accurate and were aided hugely by the novel idea of a football on top of the rocket. 25